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H.C. TIJMS

A FORMULA FOR THE LONG RUN AVERAGE COST IN SEMI-MARKOV DECISION MODELS APPLIED ON A NUMBER OF CONTINUOUS TIME INVENTORY MODELS

2e boerhaavestraat 49 amsterdam

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CONTENTS

1. Introduction

We are concerned with a semi-Markov decision model which has, roughly speaking, the following feature. Some natural process, that is, a process in which no decisions are made, can be defined, such that the decisionprocess is the superposition of the natural process and decisions made in certain states of the natural process. The advantage of the disintegration of the decisionprocess will appear to be situated in the calculation of the expected costs incurred between two successive decisions, when the decisionprocess has reached the "steady-state". The ideas underlying this approach are due to DE LEVE [11].

In section 2 the decisionmodel is defined and under rather weak conditions a formula is found for the long-run average expected cost per unit of time.

This formula is applied on a number of continuous time inventory models.

In section 3 we give some preliminary results that will be needed in the analysis of the inventory models.

In section 4 we consider a single item inventory model in which the customers arrive according to a Poisson process. The demands of the customers are mutually independent and identically distributed random variables with a geometric probability distribution and independent of the arrival process. Excess demands are lost. The ordering policy followed is an (s,S) policy of the following type. When no order is outstanding and the stock level i falls below s, then an order for S-i units is given; otherwise, no ordering is done. The numbers s and S are given integers with $s \geq 1$ and $S-s+1 \geq s$. The lead time of an order is a constant $T \geq 0$. The costs involved are ordering costs, inventory costs and lost sales costs. A formula is found for the long-run average cost per unit of time for the (s,S) policy. This formula is well-known [8] for the case in which the demand per customer equals 1. Further we consider in section 4 a variant of the (s,S) policy, where the ordering size is fixed.

In section 5 we consider again a single item (s,S) inventory model in which the customers arrive according to a Poisson process. The demands of the customers are mutually independent, positive, identically distributed random variables with a discrete probability distribution and independent of the arrival process. Excess demands are backlogged. The ordering policy followed is an (s,S) policy, which is based on the stock on hand plus an order. The numbers s and S are given integers with $s \leq S$. The lead time of an order is a constant $T \geq 0$. The costs involved are ordering costs, inventory costs and backorder costs. A formula is found for the long-run average cost per unit of time for the (s,S) policy. This formula is known [8] for the case in which the demand per customer equals 1.

Finally,in section 6 we consider a two-item invertory model in which the customers arrive according to a Poisson process. The demands of the customers are mutually independent, identically distributed random variables and independent of the arrival process. The probability that a customer demands for one unit of item j is p_j, j = 1, 2, where $p_1+p_2 = 1$. Excess demands are backlogged. The ordering policy followed is a (r_1,Q_1,r_2,Q_2) policy of the following type. When the stock on hand plus an order of item 1 and item 2 fall to i_1 and i_2 respectively, and either $i_1 = r_1$ or $i_2 = r_2$, order then simultaneously $r_1+Q_1-i_1$ units of item 1 and $r_2+Q_2-i_2$ units of item 2, otherwise, do not order. The lead time of an order is a constant $T \geq 0$. The costs involved are ordering costs, inventory costs and backorder costs. A formula is found for the long-run average cost per unit of time for the (r_1,Q_1,r_2,Q_2) policy. This formula is known for the symmetric case $p_1 = p_2$, $r_1 = r_2$, $Q_1 = Q_2$ and $T = 0$ [15].

2. Model and the long-run cost

Suppose that a stochastic process, called the natural process, can be described in the following way. At the times $\underline{\tau}_0 = 0$, $\underline{\tau}_1$, $\underline{\tau}_2$,... a system is observed and classified into some state $x \in X$, where X is a given Borel subset of some complete, separable metric space. It is assumed that $\underline{\tau}_n - \underline{\tau}_{n-1}$, n = 1, 2, ..., are mutually independent, positive, identically distributed random variables with a finite expectation. Let F(t) be the distribution function of $\underline{\tau}_{n+1} - \underline{\tau}_n$. Let $\underline{x}_n$ be the state at time $\underline{\tau}_n$. Further, it is assumed that

(a) $P\{\underline{x}_{n+1} \in A | \underline{x}_i = x_i, \underline{\tau}_{i+1} - \underline{\tau}_i = t_i, 0 \leq i \leq n\} = P\{\underline{x}_{n+1} \in A | \underline{x}_n = x_n, \underline{\tau}_{n+1} - \underline{\tau}_n = t_n\}$ for every $n \geq 0$, Borel subset A of X, $x_i \in X$, and $t_i \geq 0$, $0 \leq i \leq n$.

(b) $P\{\underline{\tau}_1 - \underline{\tau}_0 \leq t | \underline{x}_0 = x_0\} = F(t)$, $P\{\underline{\tau}_{n+1} - \underline{\tau}_n \leq t | \underline{x}_i = x_i, \underline{\tau}_{i+1} - \underline{\tau}_i = t_i, 0 \leq i \leq n\} = F(t)$ for every $n \geq 1$, $x_i \in X$ and $t_i \geq 0$, $0 \leq i \leq n$.

Let K(x,t,A) be the probability that $\underline{x}_{n+1}$ belongs to the Borel subset A of X given that $\underline{x}_n = x$ and $\underline{\tau}_{n+1} - \underline{\tau}_n = t$. Suppose that $\int_0^\infty K(x,t,A)\, F(dt)$ is well defined and is a stochastic transition function *). Hence the $\{\underline{x}_n\}$ process constitutes a discrete time Markov process, where the times between successive transitions are mutually independent random variables. We note that from renewal theory it follows that the number of transitions in any finite time interval is finite with probability one [5,16].

The following assumption will be essential in our considerations.

Assumption 1. There exists a non-empty Borel subset A_0 of X, such that

$$P\{\underline{x}_n \in A_0 \text{ for some } n \geq 0 | \underline{x}_0 = x\} = 1 \text{ for all } x \in X.$$

We suppose that a cost structure is imposed on the natural process in the following manner. When the natural process makes at time $\underline{\tau}_{n+1}$ a transition to state y, then given that $\underline{x}_n = x$ and $\underline{\tau}_{n+1} - \underline{\tau}_n = t$ a non-negative cost c(x,t,y) is incurred at time $\underline{\tau}_{n+1}$, $n \geq 1$. At time t = 0

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*) A real-valued function K(x,A), where $x \in X$ and A is a Borel subset of X, will be called a stochastic transition function if it has the following properties: (i) K(x,A) for fixed x determines a probability measure in A; (ii) K(x,A) for fixed A determines a Baire function in x.
We note that for a metric space the class of the real-valued Baire functions coincides with the class of the real-valued Borel functions (see, for instance, [7]).

no costs are incurred in the natural process.

Assume the following functions $k_0(x)$ and $t_0(x)$ are well defined and are Baire functions. For $x \notin A_0$ we define $t_0(x)$ as the expectation of the length of the time interval between t = 0 and the time at which the natural process takes on for the first time a state of A_0, and we define $k_0(x)$ as the expected cost incurred during this time interval, where x is the state on t = 0. With respect to the costs we take the time interval right closed. For $x \in A_0$ we define $k_0(x) = t_0(x) = 0$. We shall see hereafter that the functions $k_0(x)$ and $t_0(x)$ will play a fundamental part in our considerations.

Let us next describe the decisionprocess. Let I be a given Borel subset of X, such that

$$(2.1) \qquad I \supseteq A_0.$$

Let $\psi(x)$ be a given function on X with X as range too, such that

$$\psi(x) = x \qquad \text{if } x \notin I$$

and

$$\psi(x) \notin I \qquad \text{if } x \in I.$$

At the times $\underline{\tau}_0 = 0, \underline{\tau}_1, \underline{\tau}_2, \ldots$ the decisionprocess is observed and classified into some state $x \in X$. Let $\underline{z}_n$ be the state at time $\underline{\tau}_n$. The assumptions (a) and (b) with $\underline{x}_n$ replaced by $\underline{z}_n$ (see p. 3) are also imposed on the $(\underline{z}_n, \underline{\tau}_n)$ process. Hence it is assumed that $\underline{z}_{n+1}$ depends only on $\underline{z}_n$ and $\underline{\tau}_{n+1}-\underline{\tau}_n$, and that $\underline{\tau}_{n+1}-\underline{\tau}_n$ is independent of $\underline{z}_0, \ldots, \underline{z}_n$, $\underline{\tau}_1-\underline{\tau}_0, \ldots, \underline{\tau}_n-\underline{\tau}_{n-1}$.

Further, we suppose that $K(\psi(x),t,A)$ is the probability that $\underline{z}_{n+1}$ belongs to the Borel subset A of X, given that $\underline{z}_n = x$ and $\underline{\tau}_{n+1}-\underline{\tau}_n = t$. Furthermore if at time $\underline{\tau}_{n+1}$ the decisionprocess makes a transition to state y, then at time $\underline{\tau}_{n+1}$ the cost $c(\psi(x),t,y)$ is incurred given that $\underline{z}_n = x$ and $\underline{\tau}_{n+1}-\underline{\tau}_n = t$, $n \geq 0$. In addition a non-negative cost $d(x)$ is incurred in the decisionprocess at time $\underline{\tau}_n$, $n \geq 0$, when $\underline{z}_n = x$, where $d(x) = 0$ for $x \notin I$. The function $d(x)$, called the decisioncost function, is assumed to be a Baire function.

We see that the decisionprocess can be regarded as a superposition of the natural process and "the decisionmechanism $\psi(.)$". For initial state $x \notin I$, the decisionprocess behaves exactly as the natural process up to the moment that a transition occurs to state of I, say state y. By the "decisionmechanism $\psi(.)$" the state is then changed into state $\psi(y)$, this involves a cost $d(y)$, and thereafter the decisionprocess behaves exactly as a natural process with initial state $\psi(y)$ until the next moment that a transition occurs to a state of I. Note that by assumption 1 and (2.1) the return to I occurs with probability 1.

We shall next define an imbedded process of the decisionprocess. Assume from now on that on $t = 0$ the decisionprocess is in some state of I. Denote by $\underline{I}_n$ the state on the nth visit at the set I in the decisionprocess (the 0th visit is at time $t = 0$). The $\{\underline{I}_n\}$ process has I as state space. The following assumption is made about the $\{\underline{I}_n\}$ process.

Assumption 2

The $\{\underline{I}_n\}$ process is a Markov process with a stochastic transition function $p(.,.)$ (from $\underline{I}_n$ to $\underline{I}_{n+1}$) with the property that there exists a probability measure $q(.)$ on I, such that for every Borel subset A of I holds

$$(2.2) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p^{(k)}(x,A) = q(A) \qquad \text{for all } x \in I,$$

where the k-step transition function $p^{(k)}(x,A)$ is defined recursively by

$$(2.3) \qquad p^{(k)}(x,A) = \begin{cases} p(x,A) & \text{for } k = 1, \\ \int_I p(\xi,A)p^{(k-1)}(x,d\xi) & \text{for } k \geq 2. \end{cases}$$

Lemma 2.1

Let μ and μ_n, $n = 1, 2, \ldots$ be probability measures on a measureable space (Ω,F). Suppose

$$\lim_{n\to\infty} \mu_n(A) = \mu(A) \qquad \text{for every } A \in \mathcal{F}.$$

Then for any bounded measurable function f holds

$$\lim_{n\to\infty} \int_\Omega f(x)\mu_n(dx) = \int_\Omega f(x)\mu(dx).$$

This lemma is probably very wellknown. A special case of this lemma can be found in [10, p. 352]. The proof of the lemma is standard. The lemma is easily verified when f is a simple function. For an arbitrary bounded measurable function the lemma is then proved by using the fact that every bounded measurable function is the limit of a uniformly convergent sequence of simple functions.

From (2.3) it follows that for every Borel subset A of I holds

$$\frac{1}{n} \sum_{k=1}^{n} p^{(k)}(x,A) = \frac{1}{n} p(x,A) + \int_I p(\xi,A) \frac{1}{n} \sum_{k=1}^{n-1} p^{(k)}(x,d\xi),$$

$$x \in I;\ n \geq 2.$$

By this relation, assumption 2 and lemma 2.1 we have that

(2.4) $$q(A) = \int_I p(\xi,A)q(d\xi) \quad \text{for every Borel subset A of } I.$$

Moreover, it follows from assumption 2 and lemma 2.1 that for any real-valued bounded Baire function f on I holds

(2.5) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} E\{f(\underline{I}_n) | \underline{I}_0 = x\} = \int_I f(\xi)q(d\xi)$$

$$\text{for every } x \in I.$$

Given that the decisionprocess is in state $x \in I$ at time $t = 0$, let $\underline{t}(x)$ be the length of the time interval between $t = 0$ and the time at which the decisionprocess makes the next transition to a state of I, and let $\underline{k}(x)$ be the cost incurred in the decisionprocess during this time interval. We take this time interval left closed and right open with respect to the decisioncost d(.) and we take the interval left open and right closed with respect to the cost c(.,.,.).

Assumption 3

The functions $K(x) = \mathcal{E}\underline{k}(x)$ and $T(x) = \mathcal{E}\underline{t}(x)$, $x \in I$, are bounded Baire functions.

Lemma 2.2

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \mathcal{E}\{\underline{k}(\underline{I}_n)|\underline{I}_0=x\}}{\sum_{k=0}^{n} \mathcal{E}\{\underline{t}(\underline{I}_n)|\underline{I}_0=x\}} = \frac{\int_I K(\eta)q(d\eta)}{\int_I T(\eta)q(d\eta)}, \quad x \in I.$$

Proof

Since

$$\frac{1}{n}\sum_{k=0}^{n} \mathcal{E}\{\underline{k}(\underline{I}_n)|\underline{I}_0 = x\} = \frac{1}{n}\sum_{k=0}^{n} \mathcal{E}\{K(\underline{I}_n)|\underline{I}_0 = x\}, \; x \in I;\; n \geq 1,$$

and

$$\frac{1}{n}\sum_{k=0}^{n} \mathcal{E}\{t(\underline{I}_n)|\underline{I}_0 = x\} = \frac{1}{n}\sum_{k=0}^{n} \mathcal{E}\{T(\underline{I}_n)|\underline{I}_0 = x\}, \; x \in I;\; n \geq 1,$$

the lemma follows immediately by applying relation (2.5).

For any $x \in I$, let

$$k_1(x) = d(x) + k_0(\psi(x))$$

and

$$t_1(x) = t_0(\psi(x)).$$

Assumption 4

$$\int_I k_0(\eta)q(d\eta) < \infty \quad \text{and} \quad \int_I t_0(\eta)q(d\eta) < \infty.$$

Lemma 2.3

$$\int_I K(\eta)q(d\eta) = \int_I \{k_1(\eta)-k_0(\eta)\}\ q(d\eta)$$

and

$$\int_I T(\eta)q(d\eta) = \int_I \{t_1(\eta)-t_0(\eta)\}\ q(d\eta).$$

Proof

For any $x \in I$, we have

$$k_1(x) = K(x) + \int_I k_0(\xi)p(x,d\xi)$$

and

$$t_1(x) = T(x) + \int_I t_0(\xi)p(x,d\xi).$$

From Fubini's theorem and (2.4) it follows that

$$\int_I k_1(x)q(dx) = \int_I K(x)q(dx) + \int_I k_0(\xi)q(d\xi)$$

and

$$\int_I t_1(x)q(dx) = \int_I T(x)q(dx) + \int_I t_0(\xi)q(d\xi).$$

This ends the proof.

Assumption 5

There exists a state $x^* \in I$, such that

$$P(\underline{I}_n = x^* \text{ for some } n \geq 1 | \underline{I}_0 = x) = 1 \qquad \text{for every } x \in I,$$

and

$$E(\underline{N}|\underline{I}_0 = x) < \infty \qquad \text{for every } x \in I,$$

where

$$\underline{N} = \min(n|n \geq 1,\ \underline{I}_n = x^*).$$

Given that the decisionprocess is in state $x \in I$ at time $t = 0$, let $\underline{t}_c(x)$ be the length of the time interval between $t = 0$ and the time at which the decisionprocess makes a transition to state x^* for the first time (the epoch $t = 0$ is excluded), and let $\underline{k}_c(x)$ be the cost incurred in the decisionprocess during this time interval. We take this time interval left closed and right open with respect to the decisioncost d(.) and left open and right closed with respect to the costfunction c(.,.,.).

Assumption 6

$$E\underline{k}_c(x) < \infty \text{ and } E\underline{t}_c(x) < \infty \qquad \text{for every } x \in I.$$

We note that the return to state x^* in the decisionprocess is a persistent recurrent event.

Let $\underline{W}_t$ be the cumulative cost incurred in the decisionprocess during the time interval $[0,t]$. We take this interval left closed and right open with respect to the decisioncost d(.) and left open and right closed with respect to the costfunction c(.,.,.).

Theorem 2.1

$$\lim_{t\to\infty} \frac{1}{t} E(\underline{W}_t | \underline{I}_0 = x) = \frac{\alpha}{\beta} \qquad \text{for every } x \in I,$$

where

$$\alpha = \int_I K(\eta)q(d\eta) = \int_I \{k_1(\eta) - k_0(\eta)\}\, q(d\eta)$$

and

$$\beta = \int_I T(\eta)q(d\eta) = \int_I \{t_1(\eta) - t_0(\eta)\}\, q(d\eta).$$

Proof

Consider first the case $\underline{I}_0 = x^*$. Let $\underline{\nu}_0 = 0 < \underline{\nu}_1 < \underline{\nu}_2 < \ldots$ be the increasing sequence of indices n for which $\underline{I}_n = x^*$. The $\{\underline{\nu}_n\}$ process is a renewal process. For any $n \geq 0$, let $\underline{m}_n = \max\{k | \underline{\nu}_k \leq n\}$. By the elementary renewal theorem we have [5, 16]

$$(2.6) \qquad \lim_{n\to\infty} \frac{E\underline{m}_n}{n} = \frac{1}{E\underline{\nu}_1} .$$

Observe that $E\underline{\nu}_1 = E(\underline{N} | \underline{I}_0 = x^*)$, and hence $E\underline{\nu}_1$ is finite and positive.

Let $\underline{\gamma}_n$, $n \geq 1$, be the length of the time interval between the (n-1)th and the nth visit to state x^* in the decisionprocess (the 0th visit is at time $t = 0$). Observe that $\underline{\gamma}_1, \underline{\gamma}_2, \ldots$ are mutually independent, positive, identically distributed random variables with a finite expectation. Let $\underline{\delta}_n$, $n \geq 1$, be the cost incurred in the decisionprocess during the time interval between the (n-1)th and the nth visit to state x^*. We take this time interval left closed and right open with respect to the decisioncost $d(.)$ and left open and right closed with respect to the costfunction $c(.,.,.)$. Define $\underline{\gamma}_0 = 0$ and for any $t \geq 0$, let $\underline{n}_t = \max\{k \mid \underline{\gamma}_0 + \ldots + \underline{\gamma}_k \leq t\}$. By the elementary renewal theorem we have

$$(2.7) \qquad \lim_{t\to\infty} \frac{E\underline{n}_t}{t} = \frac{1}{E\underline{\gamma}_1} .$$

Since the costs are nonnegative, we have

$$(2.8) \qquad \frac{1}{t} E(\sum_{i=1}^{\underline{n}_t} \underline{\delta}_i) \leq \frac{1}{t} E(\underline{W}_t \mid \underline{I}_0 = x^*) \leq \frac{1}{t} E(\sum_{i=1}^{\underline{n}_t+1} \underline{\delta}_i), \quad t > 0.$$

Using Wald's identity *), we obtain

$$(2.9) \qquad \frac{1}{t} E(\sum_{i=1}^{\underline{n}_t+1} \underline{\delta}_i) = \frac{1}{t} E(\underline{n}_t+1)\, E\underline{\delta}_1.$$

Let

$$D(t) = E(\sum_{i=1}^{\underline{n}_t} \underline{\delta}_i) \text{ and } E(t) = \int_0^t E(\underline{\delta}_1 \mid \underline{\gamma}_1 = u)\, C(du), \; t \geq 0,$$

where $C(u)$ is the distributionfunction of $\underline{\gamma}_1$. Using a standard argument from renewal theory, we have

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*) Walds identity (see, for instance, [3,5]). Let $\{\underline{u}_n\}$, $n \geq 1$, be a sequence of mutually independent, identically distributed random variables with a finite expectation, and let $\underline{m}$ be a positive integral-valued random variable with a finite expectation. If the event $\{\underline{m} = m\}$ is independent of $\underline{u}_{m+1}, \underline{u}_{m+2}, \ldots$, for every $m \geq 1$, then

$$E(\sum_{k=1}^{\underline{m}} \underline{u}_k) = E\underline{u}_1\, E\underline{m}.$$

$$D(t) = E(t) + \int_0^t D(t-u)\, C(du), \qquad t \geq 0.$$

Applying a well known limit theorem from renewal theory [5], we obtain

$$(2.10) \qquad \lim_{t\to\infty} \frac{D(t)}{t} = \frac{E(\infty)}{E\underline{\gamma}_1} = \frac{E\underline{\delta}_1}{E\underline{\gamma}_1} .$$

From (2.7), (2.8), (2.9) and (2.10) it follows that

$$(2.11) \qquad \lim_{t\to\infty} \frac{1}{t} E(\underline{W}_t \,|\, \underline{I}_0 = x^*) = \frac{E\underline{\delta}_1}{E\underline{\gamma}_1} .$$

Observe that $E\underline{\delta}_1 = k_c(x^*)$ and $E\underline{\gamma}_1 = t_c(x^*)$.
In the same way it follows from

$$\frac{1}{n} E\Big(\sum_{i=1}^{\underline{m}_n} \underline{\delta}_i\Big) \leq \frac{1}{n} E\Big(\sum_{i=0}^{n} \underline{k}\,(\underline{I}_i) \,|\, \underline{I}_0 = x^*\Big) \leq \frac{1}{n} E\Big(\sum_{i=1}^{\underline{m}_n+1} \underline{\delta}_i\Big), \; n \geq 1$$

and

$$\frac{1}{n} E\Big(\sum_{i=1}^{\underline{m}_n} \underline{\gamma}_i\Big) \leq \frac{1}{n} E\Big(\sum_{i=0}^{n} \underline{t}(\underline{I}_i) \,|\, \underline{I}_0 = x^*\Big) \leq \frac{1}{n} E\Big(\sum_{i=1}^{\underline{m}_n+1} \underline{\gamma}_i\Big), \; n \geq 1,$$

that

$$(2.12) \qquad \lim_{n\to\infty} \frac{1}{n} E\Big(\sum_{i=0}^{n} \underline{k}(\underline{I}_i) \,|\, \underline{I}_0 = x^*\Big) = \frac{E\underline{\delta}_1}{E\underline{\nu}_1}$$

and

$$(2.13) \qquad \lim_{n\to\infty} \frac{1}{n} E\Big(\sum_{i=0}^{n} \underline{t}(\underline{I}_i) \,|\, \underline{I}_0 = x^*\Big) = \frac{E\underline{\gamma}_1}{E\underline{\nu}_1} .$$

From (2.11), (2.12) and (2.13) follows the relation *)

$$\lim_{t\to\infty} \frac{1}{t} E(\underline{W}_t \,|\, \underline{I}_0 = x^*) = \lim_{n\to\infty} \frac{E\Big(\sum_{i=0}^{n} \underline{k}(\underline{I}_i) \,|\, \underline{I}_0 = x^*\Big)}{E\Big(\sum_{i=0}^{n} \underline{t}(\underline{I}_i) \,|\, \underline{I}_0 = x^*\Big)} .$$

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*) This relation has also been noticed by ROSS [14].

Next consider the case $\underline{I}_0 = x$, where x is an arbitrary state of I. Let $W(t) = E(\underline{W}_t | \underline{I}_0 = x^*)$, then

$$\frac{1}{t}\int_0^t W(t-u)\ G(du) \leq \frac{1}{t}\ E(\underline{W}_t | \underline{I}_0 = x) \leq \frac{1}{t}\ E\underline{k}_c(x) + \frac{1}{t}\int_0^t W(t-u)\ G(du),$$

where G(u) is the distributionfunction of $\underline{t}_c(x)$. Using the fact that W(t) is nonnegative and nondecreasing for $t \geq 0$, it is standard to prove that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t W(t-u)\ G(du) = \lim_{t\to\infty} \frac{W(t)}{t}\ .$$

Hence

$$(2.15)\qquad \lim_{t\to\infty} \frac{1}{t}\ E(\underline{W}_t | \underline{I}_0 = x) = \lim_{t\to\infty} \frac{1}{t}\ E(\underline{W}_t | \underline{I}_0 = x^*), \qquad x \in I.$$

The theorem follows now from (2.14), (2.15) and the lemmas 2.2 and 2.3.

<u>Remark 2.1</u> <u>The Doeblin condition</u>

From the theory of Markov processes [3] it follows that assumption 2 is satisfied if the following conditions hold

(1) The $\{\underline{I}_n\}$ process is a Markov process with a stochastic transition function p(.,.) that satisfies the Doeblin condition, that is, there is a (finite-valued) measure m on the Borel sets of I with $m(I) > 0$, an integer $\nu \geq 1$, and a positive ε, such that for every $x \in I$ holds

$$p^{(\nu)}(x,A) \leq 1-\varepsilon \qquad \text{if } m(A) \leq \varepsilon.$$

(2) The Markov process $\{\underline{I}_n\}$ has only one ergodic set.

We note that the Doeblin condition is always satisfied if I is finite [3].

<u>Remark 2.2</u> <u>I is denumerable</u>

Consider the case that I is denumerable. Suppose that the $\{\underline{I}_n\}$

process is a Markov chain for which the state space I is a positive recurrent class. Let $p_{ij}^{(n)}$, $i,j \in I$, be the n-step transition probabilities of the Markov chain $\{\underline{I}_n\}$. Assumption 2 is now superfluous, since for any $i,j \in I$ the sequence $\{p_{ij}^{(n)}\}$, $n \geq 1$ has a Cesàrolimit independent of i, say q_j, and $\sum q_j = 1$ [2, pp. 32-33]. Furthermore, relation (2.5) and hence the lemmas 2.2 and 2.3 and theorem 2.1 remain valid when we replace assumption 3 by the weaker assumption $\sum K(j)q_j < \infty$ and $\sum T(j)q_j < \infty$ [2, p. 89]. Finally, assumption 5 is automatically satisfied [2, p. 59].

Remark 2.3 I is finite

Consider the case that I is finite. Suppose the process $\{\underline{I}_n\}$ is a Markov chain with no disjoint closed sets. From the theory of finite Markov chains it then follows that the assumptions 2 and 5 are automatically satisfied (see, for instance, [2] and [9]).

Remark 2.4 The "flexibility" in $c(x,t,y)$ and $P(x,t,y)$ for $x \in I$.

Using the fact that $\psi(x) \notin I$ for $x \in I$, it is readily seen that $K(u)$ does not depend on the values of the function $c(x,t,y)$ for $x \in I$. Consequently, the long-run average cost per unit of time for the decisionprocess is independent of $c(x,t,y)$ for $x \in I$. Furthermore, it is easy to see that $p(.,.)$, $K(x)$, $T(x)$ and hence the long-run average cost per unit of time for the decisionprocess are independent of $K(x,t,y)$, $x \in I$. This means that we may define $c(x,t,y)$ and $K(x,t,y)$ for $x \in I$ in as convenient a manner as possible, where, of course, the assumptions 1 and 4 have to be satisfied. This "flexibility" in $c(x,t,y)$ and $K(x,t,y)$ may simplify the determination of the functions $k_1(x)-k_0(x)$ and $t_1(x)-t_0(x)$.

Remark 2.5 A relation between $K(x)$ and $k_c(x^*)$, $T(x)$ and $t_c(x^*)$.

Clearly, we have

$$k_c(x) = K(x) + \int_I k_c(\xi)\, p(x,d\xi) - p(x,\{x^*\})\, k_c(x^*), \quad x \in I.$$

Using Fubini's theorem and (2.4), we obtain

$$\int_I k_c(x)q(dx) = \int_I K(x)q(dx) + \int_I k_c(\xi)q(d\xi) - q(\{x^*\})\, k_c(x^*),$$

and hence

$$(2.16) \qquad \int_I K(x)q(dx) = q(\{x^*\})\, k_c(x^*),$$

provided that $\int_I k_c(x)q(dx) < \infty$. In a similar way, we have

$$(2.17) \qquad \int_I T(x)q(dx) = q(\{x^*\})\, t_c(x^*),$$

provided that $\int_I t_c(x)q(dx) < \infty$. In order to prove that $q(\{x^*\}) > 0$, let $\tilde{\underline{z}}_n = 1$ if $\underline{I}_n = x^*$, and let $\tilde{\underline{z}}_n = 0$ if $\underline{I}_n = x$. By (2.2) and (2.6), we have

$$q(\{x^*\}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P\{\underline{I}_k = x^* | \underline{I}_0 = x^*\} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E\tilde{\underline{z}}_k =$$

$$= \lim_{n\to\infty} \frac{1}{n} E\underline{m}_n = \frac{1}{E(\underline{N}|\underline{I}_0=x^*)},$$

and hence $q(\{x^*\}) > 0$.

Using (2.11), (2.15), (2.16) and (2.17), it is readily seen that theorem 2.1 remains valid when we replace assumption 3 by the assumption that $\int K(x)q(dx) < \infty$ and $\int T(x)q(dx) < \infty$ and add to assumption 6 the assumption $\int k_c(x)q(dx) < \infty$ and $\int t_c(x)q(dx) < \infty$.

3. Preliminaries

Suppose that in the time interval $(0,\infty)$ customers arrive at a store at times $\underline{\tau}_1, \underline{\tau}_2, \ldots$, where the interarrival times $\underline{\tau}_k - \underline{\tau}_{k-1}$, $k = 1, 2, \ldots$, $(\underline{\tau}_0 = 0)$, are mutually independent, positive, identically distributed random variables with distribution function $1-e^{-\lambda t}$, i.e., the customers arrive according to a Poisson process with rate λ. Each customer demands for a single item. Let $\underline{\xi}_0 = 0$, and let $\underline{\xi}_1, \underline{\xi}_2, \ldots$ be mutually independent, nonnegative, integral-valued random variables with the common probability distribution $\phi(j) = P\{\underline{\xi}_n = j\}$, $(j \geq 0; n \geq 1)$, and independent of the arrival process. The random variable $\underline{\xi}_n$ represents the size of the demand of the nth customer. It is no restriction to assume that *)

$$\phi(0) = 0.$$

Furthermore, it is assumed that

$$\mu \overset{\text{def}}{=} \sum_{j=1}^{\infty} j\phi(j) < \infty.$$

For any $t \geq 0$, let

$$\underline{n}(t) = \max\{n | \underline{\tau}_n \leq t\}.$$

Observe that $\underline{n}(0) = 0$ with probability one. The random variable $\underline{n}(t)$ represents the number of customers arriving during $(0,t]$. We review the following well known properties of the Poisson proces $\{\underline{n}(t)\}$, $t \geq 0$, [5].

(i) The probability distribution of $\underline{n}(t)$ is given by

$$P\{\underline{n}(t) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, \dots .$$

In words, the random variable $\underline{n}(t)$ has a Poisson distribution with expectation λt.

(ii) The random variable $\underline{\tau}_{\underline{n}(t)+1} - t$ has the same exponential distribution as the $\underline{\tau}_k - \underline{\tau}_{k-1}$. In words, given an arbitrary but fixed point of time, the waiting time to the next arrival has the same distribution as the times between successive arrivals, irrespective of the "past".

Define

$$\phi^{(n)}(j) = \begin{cases} \phi(j), & j \geq 1;\ n = 1, \\ \sum_{k=1}^{j} \phi^{(n-1)}(k)\phi(j-k), & j \geq 1;\ n \geq 2. \end{cases}$$

---

*) If $\phi(0) > 0$, then the customers with a positive demand arrive according to a Poisson process with rate $\lambda(1-\phi(0))$.

We have for $n \geq 1$ that $\{\phi^{(n)}(j)\}$, $j \geq 1$, constitutes the probability distribution of $\underline{\xi}_1 + \ldots + \underline{\xi}_n$.

Let

$$m(j) = \sum_{n=1}^{\infty} \phi^{(n)}(j), \qquad j \geq 1.$$

The renewal quantity $m(j)$ can be computed from

$$(3.1) \qquad m(j) = \phi(j) + \sum_{k=1}^{j} \phi(j-k)m(k), \qquad j \geq 1.$$

For any $k \geq 0$, let

$$\underline{N}(k) = \max\{n \mid \underline{\xi}_0 + \ldots + \underline{\xi}_n \leq k\}.$$

From renewal theory it is known that [5,16]

$$E\underline{N}(k) = \sum_{j=1}^{k} m(j), \qquad k \geq 0,$$

where we adopt the convention $\sum_a^b = 0$ if $a > b$.

For any $t \geq 0$, let

$$\underline{v}(t) = \underline{\xi}_0 + \ldots + \underline{\xi}_{\underline{n}(t)}.$$

We have for $t > 0$ that $\underline{v}(t)$ is the cumulative demand in the time interval $(0,t]$.

For any $t \geq 0$, let

$$a_k(t) = P\{\underline{v}(t) = k\}, \qquad k = 0, 1, \ldots .$$

Clearly, we have for any $t \geq 0$ that

$$a_0(t) = P\{\underline{n}(t) = 0\} = e^{-\lambda t}$$

and

$$a_k(t) = \sum_{j=1}^{k} P\{\underline{n}(t) = j,\ \underline{\xi}_1 + \dots + \underline{\xi}_j = k\} =$$

$$= \sum_{j=1}^{k} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \phi^{(j)}(k), \qquad k = 1, 2, \dots .$$

It is well known that [4]

$$E\underline{v}(t) = E\underline{n}(t) \ .\ E\underline{\xi}_1 = \lambda\mu t, \qquad t \geq 0.$$

If $\eta = \mathrm{var}(\underline{\xi}_k)$ exists, then [4]

$$\mathrm{var}(\underline{v}(t)) = \mathrm{var}(\underline{\xi}_k)\ E\underline{n}(t) + \mathrm{var}(\underline{n}(t))\ (E\underline{\xi}_k)^2 = \lambda t\eta + \lambda t\mu^2.$$

We note that if $\lambda t >> 1$, then [6]

$$a_k(t) \cong \frac{1}{\sqrt{2\pi\lambda t(\eta+\mu^2)}} \exp\left[\frac{-(k-\lambda\mu t)^2}{2\lambda t(\eta+\mu^2)}\right], \quad k = 0, 1, \dots ,$$

For any $k \geq 1$, let

$$\underline{t}_k = \underline{\tau}_{\underline{N}(k-1)+1}$$

and

$$\underline{v}_k = \underline{\xi}_1 + \dots + \underline{\xi}_{\underline{N}(k-1)+1}.$$

In words, $\underline{t}_k$ is the length of the time interval from $t = 0$ up to the epoch on which the cumulative demand exceeds $k-1$ for the first time, and $\underline{v}_k$ is the cumulative demand in this time interval.

Using Wald's identity, we obtain

$$(3.3) \qquad E\underline{t}_k = E\underline{\tau}_1 \ .\ E\{\underline{N}(k-1) + 1\} = \frac{1}{\lambda}\left(1 + \sum_{j=1}^{k-1} m(j)\right), \qquad k \geq 1;$$

and

$$(3.4) \qquad E\underline{v}_k = E\underline{\xi}_1 \ .\ E\{\underline{N}(k-1) + 1\} = \mu\left(1 + \sum_{j=1}^{k-1} m(j)\right), \qquad k \geq 1.$$

For any $k \geq 1$, let

$$\text{(3.5)} \qquad \gamma_k(n) = P\{\underline{v}_k = n\}, \qquad n = k, k+1, \ldots .$$

In renewal theory $\underline{v}_k - k+1$ is called the excess random variable. Using a standard argument from renewal theory, we have

$$\text{(3.6)} \qquad \gamma_k(n) = P\{\underline{\xi}_1 = n\} + \sum_{j=1}^{\infty} \sum_{h=1}^{k-1} P\{\underline{\xi}_1 + \ldots + \underline{\xi}_j = h, \underline{\xi}_{j+1} = n-h\} =$$

$$= \phi(n) + \sum_{j=1}^{\infty} \sum_{h=1}^{k-1} \phi^{(j)}(h)\, \phi(n-h) =$$

$$= \phi(n) + \sum_{h=1}^{k-1} \phi(n-h)\, m(h), \qquad n \geq k;\ k \geq 1.$$

Let the ι-function be defined by

$$\iota(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Clearly, we have

$$\text{(3.7)} \qquad E\iota(t-\underline{t}_k) = P\{\underline{t}_k \leq t\} = P\{\underline{v}(t) \geq k\} =$$

$$= 1 - \sum_{j=0}^{k-1} a_j(t), \qquad k \geq 1;\ t \geq 0.$$

Since the arrival process is "memoryless" and independent of the demands of the customers, we have by the theorem of total expectation that

$$\text{(3.8)} \qquad E\{(\underline{t}_k-t)\ \iota(\underline{t}_k-t)\} = \sum_{j=0}^{k-1} a_j(t)\ E\underline{t}_{k-j},\ k \geq 1;\ t \geq 0.$$

From this relation and the identity

$$E(\underline{t}_k-t) = E\{(\underline{t}_k-t)\ \iota(\underline{t}_k-t)\} + E\{(\underline{t}_k-t)\ \iota(t-\underline{t}_k)\}$$

it follows that

(3.9) $$E\{(t-\underline{t}_k)\ \iota(t-\underline{t}_k)\} = \sum_{j=0}^{k-1} a_j(t)\ E\underline{t}_{k-j} - E\underline{t}_k + t, \qquad k \geq 1;\ t \geq 0.$$

Finally, we give special attention to the case that $\underline{\xi}_n$ has a geometric distribution. Suppose

$$\phi(j) = p(1-p)^{j-1}, \qquad j \geq 1,$$

where $0 < p \leq 1$. It is known that [4]

(3.10) $$\mu = \frac{1}{p}$$

and

$$\phi^{(n)}(j) = \binom{j-1}{n-1} p^n(1-p)^{j-n}, \qquad j \geq n;\ n \geq 1.$$

Furthermore, we have the known results

(3.11) $$m(j) = p, \qquad j \geq 1$$

and

(3.12) $$\gamma_k(n) = p(1-p)^{n-k}, \qquad n \geq k;\ k \geq 1.$$

We note that (3.11) can easily be verified from (3.1) by induction. The relation (3.12) follows from (3.6) and (3.11).
From (3.3), (3.4), (3.10) and (3.11) it follows that

(3.13) $$E\underline{t}_k = \frac{1}{\lambda}(1-p+kp) \qquad k \geq 1,$$

and

$$(3.14) \qquad \mathcal{E}\underline{v}_k = k + \frac{1-p}{p}, \qquad k \geq 1.$$

If $p < 1$, then the probabilities $a_k(t)$ are given by

$$a_0(t) = e^{-\lambda t}$$

and

$$a_k(t) = \lambda tp\, e^{-\lambda t} \frac{(1-p)^{k-1}}{k} L^1_{k-1}\left(\frac{-\lambda tp}{1-p}\right), \qquad k \geq 1,$$

where L^1_k is the Laguerre polynomial [1, p. 188]. The function L^1_k is explicitly given by

$$L^1_k(x) = \sum_{m=0}^{k} \binom{k+1}{k-m} \frac{(-x)^m}{m!}, \qquad k \geq 0,$$

and L^1_k satisfies the recurrence relation

$$(k+1)\, L^1_{k+1}(x) - (2k+2-x)\, L^1_k(x) + (k+1)\, L^1_{k-1}(x) = 0, \; k \geq 1.$$

If $p = 1$, then the probabilities $a_k(t)$ are clearly given by

$$(3.15) \qquad a_k(t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, \ldots .$$

We close this section by giving some properties of the Poisson distribution. Let $p(j;\alpha)$ and $P(j;\alpha)$ be defined by

$$(3.16) \qquad p(j;\alpha) = e^{-\alpha} \frac{\alpha^j}{j!}, \qquad j = 0, 1, \ldots,$$

and

$$(3.17) \qquad P(j;\alpha) = \begin{cases} \sum_{k=j}^{\infty} p(k;\alpha), & j = 0, 1, \ldots, \\ 1, & j = -1, -2, \ldots, \end{cases}$$

where α is a nonnegative number. The following properties of the $p(j;\alpha)$ and $P(j;\alpha)$ will be used in the sequel.

$$(3.18)\qquad \sum_{j=0}^{r} (r-j)\, p(j;\alpha) = r - \alpha + \sum_{j=r}^{\infty} (j-r)\, p(j;\alpha), \qquad r \geq 0,$$

$$(3.19)\qquad \sum_{j=r}^{\infty} (j-r)\, p(j;\alpha) = \sum_{j=r+1}^{\infty} P(j;\alpha), \qquad r \geq 0,$$

$$(3.20)\qquad \sum_{j=r}^{\infty} P(j;\alpha) = \alpha P(r-1;\alpha) + (1-r)\, P(r;\alpha), \qquad r \geq 0.$$

These properties together with a large number of other ones can be found in [8].

4. An (s,S) policy for a continuous time inventory model with lost sales

4.1. Introduction

Suppose that in the time interval $(0,\infty)$ customers arrive at a store at times $\underline{\tau}_1, \underline{\tau}_2, \ldots$, where $\underline{\tau}_n - \underline{\tau}_{n-1}$, $n = 1, 2, \ldots$ ($\underline{\tau}_0 = 0$), are mutually independent, positive, identically distributed random variables with the distributionfunction $1-e^{-\lambda t}$. Each customer demands for a single item. The demand of the customer arriving at time $\underline{\tau}_n$ is a positive, integral-valued random variable $\underline{\xi}_n$. Assume that $\underline{\xi}_1, \underline{\xi}_2, \ldots$ are mutually independent random variables with the common geometric probability distribution $P\{\underline{\xi}_n = j\} = p(1-p)^{j-1}$ $(j \geq 1; n \geq 1)$, where $0 < p \leq 1$, and independent of the arrival process. Demand exceeding the available stock is lost. The ordering policy followed is an (s,S) policy of the following type. When no order is outstanding and the stock on hand i falls below s, then an order of S-i units is placed; otherwise, no ordering is done. The numbers s and S are given integers with $s \geq 1$ and $S-s+1 \geq s$. Note that for this policy never more than one order is outstanding. The lead time of an order is a constant $T \geq 0$. The costs involved are ordering costs, inventory costs and lost sales costs. The ordering costs of k units are given by $h(k)$, where $h(k) \geq 0$. The costs of carrying a unit in inventory are directly proportional to the length of time for which the unit remains in inventory.

The nonnegative constant of proportionality is c_1. The cost of each lost sale is a nonnegative constant c_2, where we assume $c_2 = 0$ if $T = 0$.

In section 4.2 we shall derive a formula for the long-run average expected cost per unit of time for the (s,S) policy. This formula is well known for the case p = 1 [8]. Special attention will be given to this case. Furthermore, we shall consider the following variant of the (s,S) policy: When no order is outstanding and the stock on hand falls below $s \geq 1$, order then $Q \geq s$ units; otherwise, do not order. We note that this policy coincides with the (s,S) policy if p = 1.

4.2. The long-run average expected cost per unit of time

First we define a natural process. Let

$$X = \{(i,k,u) \mid i,k \text{ integers and } u \text{ real}; \ i \leq S,\ S-s+1 \leq k \leq S,\ 0 \leq u < T\} \cup$$

$$\cup \{i \mid i \text{ integer},\ i \leq S\}.$$

At the times $\underline{\tau}_0 = 0, \underline{\tau}_1, \underline{\tau}_2, \ldots$ the natural process is observed and classified into some state of X. For any integer i, let

$$i^+ = \max(i,0) \text{ and } i^- = -\min(i,0).$$

When the natural process is in state (i,k,u) at time $\underline{\tau}_n$, then given that $\underline{\tau}_{n+1} - \underline{\tau}_n = t$ and $\underline{\xi}_{n+1} = j$ the next state is $(i^+-j,k,u+t)$ if $u + t < T$ and $i^+ - j$ if $u + t \geq T$. When the natural process is in state i at time $\underline{\tau}_n$, then given that $\underline{\tau}_{n+1}-\underline{\tau}_n = t$ and $\underline{\xi}_{n+1} = j$ the next state is $i^+ - j$.

From this definition of the natural process it will be clear that the state is measured just after a demand has occured. The state (i,k,u) corresponds to the situation that the stock on hand is i^+, an order of k units is outstanding since t units of time and that the demand just occured involves i^- lost sales. The state i corresponds to the situation that the stock on hand is i^+ and that the demand just occurred involves i^- lost sales.

The costfunction c(x,t,y) is defined as follows for the possible combinations of x, t, y.

$$c(x,t,y)=\begin{cases} c_1ti^+ + c_2(i^+-j)^- \text{ for } x=i,\ t>0,\ y=i^+-j,\ \text{where } j\geq 1, \\ \quad \text{and for } x=(i,k,u),\ t<T-u,\ y=(i^+-j,k,u+t),\ \text{where } j\geq 1, \\ c_1ti^+ + c_1(u+t-T)k + c_2(i^++k-j)^- \\ \quad \text{for } x=(i,k,u),\ t\geq T-u,\ y=i^++k-j,\ \text{where } j\geq 1. \end{cases}$$

In words, for any unit kept in stock for a time t during the time interval $(\underline{\tau}_n,\underline{\tau}_{n+1}]$ there are incurred inventory costs c_1t at time $\underline{\tau}_{n+1}$, and for any lost sale occuring at time $\underline{\tau}_{n+1}$ there are incurred costs c_2 at time $\underline{\tau}_{n+1}$.

The natural process is now completely described. Clearly, assumption 1 is satisfied for the choice

$$A_0 = \{i \mid i \leq 0\}.$$

The "decisionset" I is defined by

$$I = \{i \mid i \leq s-1\}.$$

Observe that $I \supseteq A_0$, since $s \geq 1$. The "decisionmechanism" $\psi(i)$, $i \in I$, and the decisioncost function $d(i)$, $i \in I$, are defined by

$$\psi(i) = \begin{cases} (i,S-i,0) & \text{if } i \geq 1, \\ (0,S,0) & \text{if } i \leq 0, \end{cases}$$

and

$$d(i) = \begin{cases} h(S-i) & \text{if } i \geq 1, \\ h(S) & \text{if } i \leq 0. \end{cases}$$

We note that the so defined decisionprocess describes adequately the evolution in the (s,S) inventory model.

It is easy to see (c.f. section 3)

$$k_0(i) = \begin{cases} c_1 \sum_{j=1}^{i} E\underline{t}_j + c_2\, E(\underline{v}_i - i), & i \geq 1, \\ 0 & i \leq 0, \end{cases}$$

$$\begin{aligned} k_0((i,k,u)) = {} & c_1 \sum_{j=1}^{i^+} E\underline{t}_j + c_1 \sum_{j=0}^{i^+-1} a_j(T-u) \sum_{h=i^+-j+1}^{i^+-j+k} E\underline{t}_h \\ & + c_1 \sum_{j=i^+}^{\infty} a_j(T-u) \sum_{h=1}^{k} E\underline{t}_h + c_2 \sum_{j=i^+}^{\infty} (j-i^+) a_j(T-u) + \\ & + c_2 \sum_{j=0}^{i^+-1} a_j(T-u)\, E(\underline{v}_{i^+-j+k} - i^+ + j - k) + \\ & + c_2 \sum_{j=i^+}^{\infty} a_j(T-u)\, E(\underline{v}_k - k) \end{aligned}$$

and

$$t_0(i) = \begin{cases} E\underline{t}_i, & i \geq 1, \\ 0, & i \leq 0, \end{cases}$$

$$t_0((i,k,u)) = T-u + \sum_{j=0}^{i^+-1} a_j(T-u)\, E\underline{t}_{i^+-j+k} + \sum_{j=i^+}^{\infty} a_j(T-u) E\underline{t}_k .$$

Using (3.2), (3.13) and (3.14), we obtain after some straightforward calculations

$$(4.1)\qquad k_1(i) - k_0(i) = d(i) + k_0((i,S-i,0)) - k_0(i) =$$

$$= h(S-i) + \frac{c_1}{\lambda}\{(S-i)(1-p) + \frac{p}{2}(S-i)(S-i+1) +$$

$$+ (S-i)\, p \sum_{j=1}^{i-1} (i-j)a_j(T)\} + c_2 \sum_{j=i}^{\infty}(j-i)a_j(T)$$

$$\text{for } 1 \leq i \leq s-1.$$

$$k_1(i) - k_0(i) = h(S) + k_0((0,S,0)) =$$

$$= h(S) + \frac{c_1}{\lambda}\{S(1-p) + \frac{p}{2}\, S(S+1)\} +$$

$$+ c_2(\frac{\lambda T}{p} + \frac{1-p}{p}) \qquad \text{for } i \leq 0$$

and

$$(4.2)\qquad t_1(i) - t_0(i) = T + \frac{p}{\lambda}\sum_{j=0}^{i-1}(S-j)a_j(T) + \frac{p}{\lambda}(S-i)\sum_{j=i}^{\infty} a_j(T) - \frac{ip}{\lambda},$$

$$\text{for } 1 \leq i \leq s-1.$$

$$t_1(i) - t_0(i) = T + \frac{1}{\lambda}(1-p+pS) \qquad \text{for } i \leq 0.$$

Next we analyze the Markov chain $\{\underline{I}_n\}$. For any $i,j \in I$, let

$$p_{ij} = P\{\underline{I}_{n+1} = j \,|\, \underline{I}_n = i\}.$$

By (3.5) and (3.12) we have for any $i,j \in I$ that

$$p_{ij} = \sum_{h=0}^{i^+-1} a_h(T)\gamma_{S-h-s+1}(S-h-j) + \sum_{h=i^+}^{\infty} a_h(T)\gamma_{S-i^+-s+1}(S-i^+-j) =$$

$$= \sum_{h=0}^{i^+-1} a_h(T)p(1-p)^{s-1-j} + \sum_{h=i^+}^{\infty} a_h(T)p(1-p)^{s-j-1} =$$

$$= p(1-p)^{s-1-j}.$$

Since the probabilities p_{ij} do not depend on i the state space I of the Markov chain $\{\underline{I}_n\}$ is a positive recurrent class, and hence the asumptions 2 and 5 are satisfied (c.f remark 2.2 in section 2). Moreover, the stationary probability distribution $\{q_j\}$, $j \in I$, of the Markov chain $\{\underline{I}_n\}$ is given by

$$q_j = p(1-p)^{s-1-j}, \qquad j < s.$$

It is easy to see that the assumptions 3, 4 and 6 are satisfied. Hence for each initial state the long-run average expected cost per unit of time for the (s,S) policy is given by

$$(4.3) \qquad g = \frac{\sum_{j=-\infty}^{s-1} q_j\{k_1(j)-k_0(j)\}}{\sum_{j=-\infty}^{s-1} q_j\{t_1(j)-t_0(j)\}} = \frac{\sum_{j=1}^{s-1} p(1-p)^{s-1-j}\{k_1(j)-k_0(j)\}+(1-p)^{s-1}\{k_1(0)-k_0(0)\}}{\sum_{j=1}^{s-1} p(1-p)^{s-1-j}\{t_1(j)-t_0(j)\}+(1-p)^{s-1}\{t_1(0)-t_0(0)\}},$$

where $k_1(j) - k_0(j)$ and $t_1(j) - t_0(j)$ are given by (4.1) and (4.2).

Now consider the special case

$$p = 1,$$

i.e., each customer demands one unit. The (s,S) policy now becomes the familiar (r,Q) policy, where $r = s-1$ and $Q = S-s+1$, that is, when no order is outstanding and the stock on hand reaches the reorder level r, order then Q units; otherwise, do not order.

Using (3.15), (3.16), (3.18) and the fact that $q_r = 1$, we find after some straighforward calculations that the long-run average expected cost per unit of time for the (r,Q) policy is given by

$$a(r,Q) = \frac{h(Q)+\frac{c_1Q}{\lambda}\{r-\lambda T+\frac{Q+1}{2}\}+(\frac{c_1Q}{\lambda}+c_2)\sum_{j=r}^{\infty}(j-r)p(j;\lambda T)}{\frac{Q}{\lambda}+\frac{1}{\lambda}\sum_{j=r}^{\infty}(j-r)p(j;\lambda T)}$$

This formula is well known [8].

Remark 4.1

Suppose the integers $r^* \geq 1$ and $Q^* \geq r^*+1$ minimize $a(r,Q)$, where $r \geq 0$, $Q \geq \max(r,1)$ and r, Q integers. Assume $a(r^*,Q^*) < \lambda c_2$. In practical situations we will have that $a(r^*,Q^*) < \lambda c_2$, since λc_2 is the long-run average cost per unit of time for the policy which prescribes to hold no stock. We have the necessary conditions

$$(4.4) \qquad a(r^*\pm 1,Q^*) \geq a(r^*,Q^*) \text{ and } a(r^*,Q^*\pm 1) \geq a(r^*,Q^*).$$

Using (3.19) and (3.20), we find after some straighforward calculations that (4.4) can be written as [12]

$$P(r^*+1;\lambda T) \leq \frac{c_1Q^*}{c_1Q^*+\lambda c_2-a(r^*,Q^*)} \leq P(r^*;\lambda T),$$

and

$$\lambda h(Q^*)-\lambda h(Q^*-1)+c_1Q^* \leq a(r^*,Q^*)-c_1\{r^*-\lambda T+\lambda TP(r^*;\lambda T)-r^*P(r^*+1;\lambda T)\} \leq$$

$$\leq \lambda h(Q^*+1)-\lambda h(Q^*)+c_1(Q^*+1).$$

Furthermore, we note that the formula for $a(r,Q)$ can be written as

$$(4.5) \qquad a(r,Q) = \frac{1}{Q}[\lambda h(Q)+c_1Q\{r-\lambda T+\frac{Q+1}{2}\} + $$

$$+ \{c_1Q+\lambda c_2-a(r,Q)\}\sum_{j=r}^{\infty}(j-r)p(j;\lambda T)].$$

When

$$h(Q) = hQ+K, \qquad Q \geq 1,$$

then from (4.5) and the necessary conditions $a(r^*,Q^*\pm 1) \geq a(r^*,Q^*)$ we obtain after some straightforward calculations [12]

$$\frac{1}{Q^*(Q^*+1)} [\lambda K+\{\lambda c_2-a(r^*,Q^*)\} \sum_{j=r^*}^{\infty}(j-r^*)p(j;\lambda T)] \leq \frac{c_1}{2} \leq$$

$$\leq \frac{1}{Q^*(Q^*-1)} [\lambda K+\{\lambda c_2-a(r^*,Q^*)\} \sum_{j=r^*}^{\infty}(j-r^*)p(j;\lambda T)],$$

and hence

$$Q^* \simeq \sqrt{\frac{2\lambda K}{c_1} + \frac{2}{c_1} \{\lambda c_2-a(r^*,Q^*)\} \sum_{j=r^*}^{\infty} (j-r^*)p(j;\lambda T)} \; .$$

Remark 4.2

Consider the following variant of the (s,S) policy. When no order is outstanding and the stock on hand falls below s, order then Q units; otherwise, do not order. The numbers s and Q are given integers with $Q \geq s \geq 1$. In the same way as (4.3) has been derived, we find that the long-run average expected cost per unit of time for this (s,Q) policy is given by

$$g = \frac{\sum_{j=1}^{s-1} p(1-p)^{s-1-j}\, k(j)+(1-p)^{s-1}\, k(0)}{\sum_{j=1}^{s-1} p(1-p)^{s-1-j}\, t(j)+(1-p)^{s-1}\, t(0)},$$

where

$$k(j)=h(Q)+ \frac{c_1 Q}{\lambda} (1-p)+ \frac{c_1 p}{2\lambda} Q(Q+1)+ \frac{c_1 Qp}{\lambda} \sum_{h=0}^{j-1} (j-h)a_h(T) +$$

$$+ c_2 \sum_{h=j}^{\infty} (h-j)a_j(T) + c_2 (\frac{1-p}{p})(1-\delta(j)), \quad 0 \leq j < s$$

and

$$t(j) = T + \frac{1-p}{\lambda} + \frac{Qp}{\lambda} + \frac{p}{\lambda}\sum_{h=0}^{j-1} (j-h)a_h(T) - \frac{1}{\lambda}(1-p+jp)\delta(j)$$

for $0 \leq j \leq s$,

where $\delta(0) = 0$ and $\delta(j) = 1$ for $j \geq 1$.

5. An (s,S) policy for a continuous time inventory model with backlogging

5.1. Introduction

Suppose that in the time interval $(0,\infty)$ customers arrive at a store at times $\underline{\tau}_1, \underline{\tau}_2, \ldots$, where $\underline{\tau}_n - \underline{\tau}_{n-1}$, $n = 1, 2, \ldots$ $(\underline{\tau}_0 = 0)$ are mutually independent, positive, identically distributed random variables with the distributionfunction $1-e^{-\lambda x}$. Each customer demands for a single item. The demand of the customer arriving at time $\underline{\tau}_n$ is a positive, integral-valued random variable $\underline{\xi}_n$. Assume $\underline{\xi}_1, \underline{\xi}_2, \ldots$ are mutually independent random variables with the common probability distribution $\phi(j) = P\{\underline{\xi}_n = j\}$, $(j \geq 1; n \geq 1)$, and independent of the arrival process. Suppose

$$\mu = \sum_{j=1}^{\infty} j\phi(j) < \infty.$$

Excess demand is backlogged. Hence the stock on hand may take on negative values. The ordering policy followed is an (s,S) policy of the following type. When the stock on hand plus on order i falls below s, then S-i units are ordered; otherwise, no ordering is done. The numbers s and S are given integers with $S \geq s \geq 1$. The lead time of an order is a constant $T \geq 0$.

The costs involved are ordering costs, inventory costs and backorder costs. The cost of ordering k units is $K\delta(k) + ck$, where $K \geq 0$, $c \geq 0$, $\delta(0) = 0$ and $\delta(k) = 1$ for $k \geq 1$. It is no restriction to assume that $c = 0$ *). The costs of carrying a unit in inventory are directly proportional to the length of time for which the unit remains in inventory. The constant of proportionality is $c_1 \geq 0$. For each unit backordered there is a fixed cost $c_2 \geq 0$ plus a nonnegative, variable cost c_3t which depends on the length of time t for which the backorder exists. Each unit backordered is delivered subsequently on the moment that stock becomes available. Observe that since $s \geq 1$, the backorder costs of a unit backordered never exceed c_2+c_3T. We assume $c_2 = c_3 = 0$ if $T = 0$.

In the next section we shall determine a formula for the long-run average expected cost per unit of time for the (s,S) policy. This formula is well known [8] for the case that $\phi(1) = 1$, i.e., each customer demands one unit.

5.2. The long-run average expected cost per unit of time

First we define a natural process. Let

$$X = \bigcup_{m=0}^{\infty} X_m,$$

where

$$X_0 = \{i \mid i \text{ integer}, i \leq S\},$$

and

$$X_m = \{(i,i_1,u_1,\ldots,i_m,u_m) \mid i,i_1,\ldots,i_m \text{ integers}; u_1,\ldots,u_m \text{ reals};$$

$$i_1,\ldots,i_m \geq 1,\ i+i_1+\ldots+i_m \leq S,\ T>u_1>u_2>\ldots>u_m \geq 0\}$$

$$\text{for } m = 1, 2, \ldots .$$

*) Since all demand is satisfied ultimately in the (s,S) inventory process, we have that the long-run average expected linear purchase costs per unit of time are equal to $c\lambda\mu$.

At the times $\underline{\tau}_0 = 0$, $\underline{\tau}_1$, $\underline{\tau}_2$, ... the natural process is observed and classified into some state $x \in X$. When the natural process is in state i at time $\underline{\tau}_n$, then given that $\underline{\tau}_{n+1}-\underline{\tau}_n = t$ and $\underline{\xi}_{n+1} = j$ the next state is i-j. When the natural process is in state $(i,i_1,u_1,\dots,i_m,u_m)$ at time $\underline{\tau}_n$, then given that $\underline{\tau}_{n+1}-\underline{\tau}_n = t$ and $\underline{\xi}_{n+1} = j$ the next state is $(i-j,i_1,u_1+t,\dots,i_m,u_m+t)$ if $t < T-u_1$,
$(i+i_1+\dots+i_h-j,i_{h+1},u_{h+1}+t,\dots,i_m,u_m+t)$ if $T-u_h \leq t < T-u_{h+1}$,
$h = 1, \dots, m-1$, and $i+i_1+ \dots + i_m-j$ if $t \geq T-u_m$.

We note that the state of the natural process is measured just after a demand has occurred. The state i corresponds to the situation that the stock on hand is i and no orders are outstanding. The state $(i,i_1,u_1,\dots,i_m,u_m)$ corresponds to the situation that the stock on hand is i and m orders are outstanding simultaneously, where the hth order has size i_h and is outstanding since u_h units of time, $h = 1, \dots, m$. Furthermore, we note that in the natural process no orders are placed, but orders already outstanding in the initial state of the natural process are delivered in the course of the natural process.

We shall define the cost function $c(x,t,y)$ verbally. For any unit kept in stock for some time t during the time interval $(\underline{\tau}_n,\underline{\tau}_{n+1}]$ in the natural process, there are incurred inventory costs c_1t at time $\underline{\tau}_{n+1}$. When in the natural process at time $\underline{\tau}_{n+1}$ a backorder arises, then for the unit backordered there are incurred at time $\underline{\tau}_{n+1}$ backorder costs c_2+c_3t if the unit backordered will be satisfied by a future delivery in the natural process, which arrives t units of time hence and backorder costs c_2+c_3T if the unit backordered is not satisfied in the natural process by a future delivery. By this description the function $c(x,t,y)$ is defined unambiguously. However we omit the formula for $c(x,t,y)$, since this formula is rather comprehensive and is not explicitly needed in the sequel.

For $x \in X$, let

$$e(x) = \begin{cases} i & \text{if } x = i \\ i + i_1 + \dots + i_m & \text{if } x = (i,i_1,u_1,\dots,i_m,u_m). \end{cases}$$

Assumption 1 is clearly satisfied for the choice

$$A_0 = \{x|e(x) \le 0\}.$$

The decisionset I is defined by

$$I = \{x|e(x) \le s-1\}.$$

Observe that $I \supseteq A_0$, since $s \ge 1$. The decisionmechanism $\psi(x)$, $x \in I$, and the decisioncost function $d(x)$, $x \in I$, are defined by

$$\psi(x) = \begin{cases} (i,S-i,0) & \text{if } x = i, \\ (i,i_1,u_1,\dots,i_m,u_m,S-e(x),0) & \text{if } x = (i,i_1,u_1,\dots,i_m,u_m), \end{cases}$$

and

$$d(x) \equiv K.$$

The so defined decisionprocess adequately describes the (s,S) inventory process.

It is easy to see that in the decisionprocess the times between successive visits to the set I are mutually independent, positive, identically distributed random variables with the same distribution as the random variable $\underline{t}_{S-s+1}$.

We shall now prove that the process $\{\underline{I}_n\}$ satisfies the assumptions 2 and 5. Let

$$\rho = P\{\underline{t}_{S-s+1} \ge T\}$$

and let

$$\rho_i = P\{\underline{v}_{S-s+1} = S-i,\ \underline{t}_{S-s+1} \ge T\}, \qquad i < s.$$

Clearly,

$$\sum_{i<s} \rho_i = \rho > 0.$$

We shall now show that the stochastic transitionfunction $p(.,.)$ of the Markov process $\{\underline{I}_n\}$ satisfies the Doeblin condition. Clearly, we have

$$(5.1) \qquad p(x,\{i\}) = \rho_i \qquad \text{for all } x \in I;\ i < s.$$

Define for any Borel subset A of I

$$m(A) = \sum_{i \in A} \rho_i .$$

Clearly, m is a finite-valued measure on the Borel sets of I with $m(I) = \rho > 0$. Let

$$\varepsilon = \frac{\rho}{2} .$$

Let A be a Borel subset of I, such that

$$m(A) \leq \varepsilon.$$

Then it follows from

$$\sum_{i \notin A} \rho_i = \rho - m(A) \geq \rho - \varepsilon = \varepsilon$$

that

$$p(x,A) \leq 1 - \sum_{i \notin A} \rho_i \leq 1 - \varepsilon.$$

Hence the stochastic transition function $p(.,.)$ satisfies the Doeblin condition (see remark 2.1). Further, it follows from (5.1) and the fact that $\rho_i > 0$ for at least one $i < s$ that the Markov process has only one ergodic set. Hence assumption 2 is satisfied. Moreover, it follows from (5.1) that the assumption 5 is satisfied. We can take for x^* any state $i < s$ with $\rho_i > 0$; if $x^* = i$, where $\rho_i > 0$, then for any initial state $x \in I$ the random variable $\underline{N}$ has a geometric probability distribution with expectation $1/\rho_i$.

We have already noted that in the decisionprocess the times between successive visits to the set I are mutually independent random variables with the same distribution as $\underline{t}_{S-s+1}$, and hence for any $x \in I$ the random variable $\underline{t}(x)$ is distributed as $\underline{t}_{S-s+1}$. Hence the denumerator β of the criterion is given by (c.f. theorem 2.1)

$$\beta = E\underline{t}_{S-s+1}. \tag{5.2}$$

Next we shall determine $k_1(x)-k_0(x)$, $x \in I$. We shall see that the function $k_1(x)-k_0(x)$ depends only on $e(x)$. Some reflection shows that in $k_1(x)$ and $k_0(x)$ the same term appears for the expected inventory costs for the $e(x)$ units which represent state x. Further, we have that in $k_1(x)$ and $k_0(x)$ the same term appears for the expected backorder costs for the $e(x)$ units which represent x. When $e(x) \geq 1$, then the expected backorder costs

$$(c_2+c_3T)\ E(\underline{v}_{e(x)}-e(x)) \tag{5.3}$$

appear in $k_0(x)$ but not in $k_1(x)$. In $k_1(x)$ there appears the term

$$c_1 \sum_{k=\max(e(x),0)+1}^{S} E\{(\underline{t}_k-T)\ \iota(\underline{t}_k-T)\} \tag{5.4}$$

for the expected inventory costs for the $S-e(x)$ units of the order placed in state x and further, there appears in $k_1(x)$ the term

$$\sum_{k=\max(e(x),0)+1}^{S} E\{(c_2+c_3(T-\underline{t}_k))\ \iota(T-\underline{t}_k)\} \tag{5.5}$$

for the expected backorder costs for the $S-e(x)$ units of the order placed in state x. Furthermore, the expected backorder costs

$$(c_2+c_3T)\ E(\underline{v}_S-S) \tag{5.6}$$

appear in $k_1(x)$ but not in $k_0(x)$.

It is now readily seen that

$$k_1(x)-k_0(x) = \begin{cases} K+(5.4)+(5.5)+(5.6)-(5.3) & \text{if } e(x) \geq 1, \\ K+(5.4)+(5.5)+(5.6) & \text{if } e(x) \leq 0. \end{cases}$$

Using (3.7), (3.8) and (3.9), we obtain after some straightforward calculations

$$(5.7) \qquad k_1(x)-k_0(x) = K + \sum_{k=i+1}^{S} [(c_1+c_3) \sum_{j=0}^{k-1} a_j(T) E\underline{t}_{k-j} + c_3(T-E\underline{t}_k) + c_2(1 - \sum_{j=0}^{k-1} a_j(T))] + (c_2+c_3T)(E\underline{v}_S-E\underline{v}_i-S+i) \qquad \text{if } e(x) = i \geq 1,$$

and

$$(5.8) \qquad k_1(x)-k_0(x) = K + \sum_{k=1}^{S} [(c_1+c_3) \sum_{j=0}^{k-1} a_j(T) E\underline{t}_{k-j} + c_3(T-E\underline{t}_k) + c_2(1- \sum_{j=0}^{k-1} a_j(T))] + (c_2+c_3T)(E\underline{v}_S-S) \qquad \text{if } e(x) = i \leq 0.$$

It is readily seen that the assumptions 3, 4 and 6 are satisfied.

For any $j < s$, let

$$\hat{A}_j = \{x | x \in I, e(x) = j\}.$$

Clearly, we have for any $n \geq 1$ that

$$P\{\underline{I}_n \in \hat{A}_j | \underline{I}_{n-1} = x\} = P\{e(\underline{I}_n) = j | \underline{I}_{n-1} = x\} = \gamma_{S-s+1}(S-j) \qquad \text{for all } x \in I;\ j < s.$$

Since

$$q(\hat{A}_j) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P\{\underline{I}_k \in \hat{A}_j | \underline{I}_0 = x\}, \qquad x \in I;\ j < s,$$

it follows that

$$(5.9) \qquad q(\hat{A}_j) = \gamma_{S-s+1}(S-j), \qquad j < s.$$

It follows now from theorem 2.1, (5.2) and (5.9) that the long-run average expected cost per unit of time for the (s,S) policy equals

$$(5.10) \qquad g = \frac{1}{E\underline{t}_{S-s+1}} \left[\sum_{i<s} \{k_1(i) - k_0(i)\} \gamma_{S-s+1}(S-i) \right],$$

where $k_1(i)-k_0(i)$ is given by (5.7) and (5.8).

Consider now the special case

$$\phi(1) = 1,$$

i.e., each customer demands one unit. The (s,S) policy now becomes the familiar (r,Q) policy, where $r = s-1$ and $Q = S-s+1$, i.e., when the stock on hand plus on order reaches the recorder level r, order then Q units; otherwise, do not order. We now have (c.f. (3.13), (3.14), (3.15) and (3.16))

$$\gamma_{S-s+1}(S-s) = 1;\ a_k(T) = p(k;\lambda T),\ k \geq 0;\ E\underline{t}_k = \frac{k}{\lambda} \text{ and } \underline{v}_k = k,\ k \geq 1.$$

After some straightforward calculations it follows from (5.7), (5.8) and (5.10) that the long-run average expected cost per unit of time for the (r,Q) policy is given by

$$a(r,Q) = \frac{\lambda K}{Q} + \frac{(c_1+c_3)}{Q} \sum_{k=r+1}^{r+Q} \sum_{j=0}^{k} (k-j)p(j;\lambda T) + c_3(\lambda T - r - \frac{(Q+1)}{2}) +$$

$$+ \frac{\lambda c_2}{Q} \sum_{k=r+1}^{r+Q} P(k;\lambda T).$$

Using (3.18) we can write a(r,Q) in the equivalent form

$$\text{(5.11)}\quad a(r,Q) = \frac{\lambda K}{Q} + c_1\left(r-\lambda T+\frac{Q+1}{2}\right) + \left(\frac{c_1+c_3}{Q}\right)\sum_{i=1}^{Q}\sum_{j=r+i+1}^{\infty} P(j;\lambda T) + \frac{\lambda c_2}{Q}\sum_{i=1}^{Q} P(r+i;\lambda T).$$

This formula can also be found in [8].

<u>Remark 5.1</u>

Suppose the integers $r^* \geq 1$ and $Q^* \geq 2$ minimize a(r,Q), where $r \geq 0$, $Q \geq 1$ and r, Q integers. We then have

$$\text{(5.12)}\qquad a(r^*\pm 1,Q^*) \geq a(r^*,Q^*) \text{ and } a(r^*,Q^*\pm 1) \geq a(r^*,Q^*).$$

Using (5.11), we find after some straightforward calculations that the necessary conditions (5.12) can be written as [12]

$$\frac{1}{Q^*}\sum_{i=1}^{Q^*} P(r^*+i+1;\lambda T) + \frac{\lambda c_2}{(c_1+c_3)Q^*}\sum_{i=1}^{Q^*} p(r^*+i;\lambda T) \leq \frac{c_1}{c_1+c_3} \leq$$

$$\leq \frac{1}{Q^*}\sum_{i=1}^{Q^*} P(r^*+i;\lambda T) + \frac{\lambda c_2}{(c_1+c_3)Q^*}\sum_{i=1}^{Q^*} p(r^*+i-1;\lambda T)$$

and

$$\frac{K}{Q^*(Q^*+1)} + \frac{(c_1+c_3)}{\lambda Q^*(Q^*+1)}\sum_{i=1}^{Q^*} iP(r^*+i+1;\lambda T) + \frac{\lambda c_2}{Q^*+1} P(r^*+Q^*+1;\lambda T) +$$

$$- \frac{\lambda c_2}{Q^*(Q^*+1)}\sum_{i=1}^{Q^*} P(r^*+i;\lambda T) \leq \frac{c_1}{2} \leq$$

$$\leq \frac{K}{Q^*(Q^*-1)} + \frac{(c_1+c_3)}{\lambda Q^*(Q^*-1)}\sum_{i=1}^{Q^*-1} iP(r^*+i+1;\lambda T) + \frac{\lambda c_2}{Q^*} P(r^*+Q^*;\lambda T) +$$

$$- \frac{\lambda c_2}{Q^*(Q^*-1)}\sum_{i=1}^{Q^*-1} P(r^*+i;\lambda T).$$

From the latter inequalities it follows that

$$Q^* \simeq \sqrt{\frac{2\lambda K}{c_1} + \frac{2\lambda}{c_1} R(r^*,Q^*)},$$

where

$$R(r^*,Q^*) = \left(\frac{c_1+c_3}{\lambda}\right) \sum_{i=1}^{Q^*} iP(r^*+i+1;\lambda T) +$$

$$+ \lambda c_2\{Q^* P(r^*+Q^*+1;\lambda T) - \sum_{i=1}^{Q^*} P(r^*+i;\lambda T)\}.$$

Observe that $R(r^*,Q^*)$ is nonnegative, since $P(k;\lambda T)$ is nonincreasing in k.

6. An (r_1,Q_1,r_2,Q_2) joint ordering policy for a two-item continuous time inventory model with backlogging

Suppose that in the time interval $(0,\infty)$ customers arrive at a store at times $\underline{\tau}_1$, $\underline{\tau}_2$, ..., where $\underline{\tau}_n-\underline{\tau}_{n-1}$, n = 1, 2, ... ($\underline{\tau}_0=0$) are mutually independent, positive, identically distributed random variables with the distribution function $1-e^{-\lambda t}$. Each customer demands either for item 1 or for item 2. The demands of the customers are mutually independent, identically distributed random variables and independent of the arrival process. The probability that a customer demands one unit of item j is p_j, j = 1, 2, where $p_1+p_2 = 1$. Denote by $\underline{\tau}_{jn}$ the time at which the nth demand for item i occurs, j = 1, 2. It is known from the theory of Poisson processes that for fixed j = 1, 2 the random variables $\underline{\tau}_{jn}-\underline{\tau}_{j,n-1}$, n = 1, 2, ... ($\underline{\tau}_{j0}=0$) are mutually independent, positive random variables with the common distribution function $1-e^{-\lambda p_j t}$. Moreover, the sequences $\{\underline{\tau}_{1n}-\underline{\tau}_{1,n-1}\}$, $n \geq 1$, and $\{\underline{\tau}_{2n}-\underline{\tau}_{2,n-1}\}$, $n \geq 1$, are mutually independent.

Excess demand is backlogged. The ordering policy followed is a (r_1,Q_1,r_2,Q_2) policy of the following type. When the stock on hand plus on order of item 1 and item 2 fall to i_1 and i_2 respectively, and either $i_1 = r_1$ or $i_2 = r_2$, order then simultaneously $r_1+Q_1-i_1$ units of item 1 and $r_2+Q_2-i_2$ units of item 2; otherwise, do not order. The numbers r_j and Q_j are given integers with $r_j \geq 0$, $Q_j \geq 1$, $j = 1, 2$. The lead time of an order is a constant $T \geq 0$.

The costs involved are ordering costs, inventory costs and backorder costs. The cost of ordering simultaneously k_1 units of item 1 and k_2 units of item 2 are given by $c_1k_1+c_2k_2+K\delta(k_1,k_2)$, where c_1, c_2, $K \geq 0$ and $\delta(k_1,k_2)$ is a given function with $0 \leq \delta(k_1,k_2) \leq 1$. It is no restriction to assume that $c_1 = c_2 = 0$ (the linear purchase cost c_j contributes $c_j\lambda p_j$ to the long-run average cost). The costs of carrying a unit of item j in inventory are directly proportional to the length of time for which the unit remains in inventory. The nonnegative constant of proportionality is c_{j1}, $j = 1, 2$. For each backorder of item j there is a fixed, nonnegative cost c_{j2} plus a variable, nonnegative cost $c_{j3}t$ which depends on the length of time t for which the backorder exists, where $c_{j2} = c_{j3} = 0$ if $T = 0$, $j = 1, 2$. Note that any backorder in the inventory process is satisfied by an order which is already outstanding on the moment the backorder arises, because r_1, $r_2 \geq 0$ and each customer demands one unit.

In the next section we shall derive a formula for the long-run average expected cost per unit of time for the (r_1,Q_1,r_2,Q_2) policy.

6.2. The long-run average expected cost per unit of time

First we define a natural process. Let

$$X = \bigcup_{m=0}^{\infty} X_m,$$

where

$$X_0 = \{(i_1,i_2) | i_1,i_2 \text{ integers, } i_1 \leq r_1 + Q_1,\ i_2 \leq r_2 + Q_2\}$$

and

$$X_m = \{((i_1,i_2),(i_{11},i_{21}),u_1,\ldots,(i_{1m},i_{2m}),u_m) | i_1,i_2,i_{jh} \text{ integers,}$$

$$i_{1h}+i_{2h} \geq 1 \text{ for } j=1,2 \text{ and } h=1,\ldots,m;\ i_j+i_{j1}+\ldots+i_{jm} \leq r_j+Q_j$$

$$\text{for } j=1,2;\ u_h \text{ reals for } h=1,\ldots,m;\ T>u_1\ \ldots>u_m \geq 0\}$$

$$\text{for } m = 1, 2, \ldots .$$

At the times $\underline{\tau}_0 = 0$, $\underline{\tau}_1$, $\underline{\tau}_2$, ... the natural process is observed and classified into some state of X. When the natural process is in state (i_1,i_2) at time $\underline{\tau}_n$, then given that $\underline{\tau}_{n+1}-\underline{\tau}_n = t$ and at time $\underline{\tau}_{n+1}$ the demand for item j is k_j, j = 1, 2, the next state is (i_1-k_1,i_2-k_2), where k_1, k_2 are 0 or 1 and $k_1+k_2 = 1$. When the natural process is in state $((i_1,i_2),(i_{11},i_{21}),u_1,\ldots,(i_{1m},i_{2m}),u_m)$ at time $\underline{\tau}_n$, then given that $\underline{\tau}_{n+1}-\underline{\tau}_n = t$ and at time $\underline{\tau}_{n+1}$ the demand for item j is k_j, j = 1, 2, the next state is $((i_1-k_1,i_2-k_2),(i_{11},i_{21}),u_1+t,\ldots,(i_{1m},i_{2m}),u_m+t)$ if $t < T-u_1$, $((i_1+i_{11}+\ldots+i_{1h}-k_1,i_2+i_{21}+\ldots+i_{2h}-k_2), (i_{1,h+1},i_{2,h+1}), u_{h+1}+t,\ldots,(i_{1m},i_{2m}),u_m+t))$ if $T-u_h \leq t < T-u_{h+1}$, h = 1, ..., m-1, and $(i_1+i_{11}+\ldots i_{1m}-k_1,i_2+i_{21}+\ldots+i_{2m}-k_2)$ if $t \geq T-u_m$.

The interpretation of the state is analogous to the interpretation of the state defined in section 5.

We shall define again the cost function $c(x,t,y)$ verbally. For any unit of item j kept in stock for some t during the time interval $(\underline{\tau}_n,\underline{\tau}_{n+1}]$ there are incurred inventory costs $c_{j1}t$ at time $\underline{\tau}_{n+1}$, j = 1, 2. When in the natural process at time $\underline{\tau}_{n+1}$ a backorder of item j arises, then for the backorder there are incurred at time $\underline{\tau}_{n+1}$ backorder costs $c_{j2}+c_{j3}t$ if the backorder will be satisfied in the natural process by a future delivery which arrives t units of time hence, and backorder costs zero if the backorder is not satisfied in the natural process by a future delivery, j = 1, 2. We note that from the definition of I it will appear that if in the natural process at time $\underline{\tau}_{n+1}$ a backorder arises which cannot be satisfied by a future delivery, then the state at time $\underline{\tau}_n$ belongs to I. Hence we may define in the natural process the backorder costs zero for a backorder which cannot be satisfied by a future delivery (c.f. remark 2.4).

For any state $x \in X$, let

$$e(x) = (e_1(x),e_2(x)),$$

where for j = 1, 2,

$$e_j(x) = \begin{cases} i_j & \text{if } x=(i_1,i_2) \\ i_j+i_{j1}+\ldots+i_{jm} & \text{if } x=((i_1,i_2),(i_{11},i_{21}),u_1,\ldots,(i_{1m},i_{2m}),u_m) \end{cases}$$

Assumption 1 is clearly satisfied for the choice

$$A_0 = \{x|e_j(x) \leq 0 \text{ for } j = 1, 2\}.$$

The "decisionset" I is defined by

$$I = \{x|e_1(x) \leq r_1 \text{ or } e_2(x) \leq r_2\}.$$

Observe that $I \supseteq A_0$. The decisionmechanism $\psi(x)$, $x \in I$, and the decisioncost function $d(x)$, $x \in I$, are defined by

$$\psi(x) = \begin{cases} (r_1+Q_1-i_1,r_2+Q_2-i_2) & \text{if } x = (i_1,i_2), \\ ((i_1,i_2),(i_{11},i_{21}),u_1,\ldots,(i_{1m},i_{2m}),u_m,(r_1+Q_1-e_1(x),r_2+Q_2-e_2(x)),0) & \\ \qquad \text{if } x = ((i_1,i_2),(i_{11},i_{21}),u_1,\ldots,(i_{1m},i_{2m}),u_m), \end{cases}$$

and

$$d(x) = K\delta(r_1+Q_1-e_1(x),r_2+Q_2-e_2(x)).$$

In the same way as in section 5 it can be verified that the assumptions 2 and 5 are satisfied. Clearly, we have for every $x \in I$ and $n \geq 0$ that

$$(6.1) \qquad P\{e(\underline{I}_{n+1}) = (r_1+k,r_2)|\underline{I}_n = x\} = p_{1k}, \; k = 1, \ldots, Q_1,$$

and

$$(6.2) \qquad P\{e(\underline{I}_{n+1}) = (r_1,r_2+k)|\underline{I}_n = x\} = p_{2k}, \; k = 1, \ldots, Q_2,$$

where

$$(6.3) \qquad p_{1k} = \binom{Q_1+Q_2-k-1}{Q_2-1} p_1^{Q_1-k} \, p_2^{Q_2}, \quad k = 1, \ldots, Q_1,$$

and

$$(6.4) \qquad p_{2k} = \binom{Q_1+Q_2-k-1}{Q_1-1} p_1^{Q_1} \, p_2^{Q_2-k}, \quad k = 1, \ldots, Q_2.$$

Observe that

$$\sum_{k=1}^{Q_1} p_{1k} + \sum_{k=1}^{Q_2} p_{2k} = 1.$$

It is easy to see that in the decisionprocess the times between successive visits to the set I are mutually independent, identically distributed random variables. For any $x \in I$, the random variable $\underline{t}(x)$ has the same distribution as $\underline{\tau}_{\underline{m}}$, where $\underline{m}$ is a random variable which is independent of $\underline{\tau}_n - \underline{\tau}_{n-1}$, $n = 1, 2, \ldots$, and has the probability distribution

$$P\{\underline{m} = k\} = p_{1,k-Q_2} + p_{2,k-Q_1}, \qquad k = \min(Q_1,Q_2), \ldots, Q_1+Q_2-1,$$

where we define $p_{1k} = p_{2k} = 0$ for $k \leq 0$. Since

$$E\underline{\tau}_{\underline{m}} = E\underline{\tau}_1 . E\underline{m} = \frac{1}{\lambda} \sum_{k=\min(Q_1,Q_2)}^{Q_1+Q_2-1} k(p_{1,k-Q_2} + p_{2,k-Q_1}),$$

we have (c.f. theorem 2.1)

$$(6.5) \qquad \beta = \sum_{k=1}^{Q_1} (Q_1+Q_2-k) \, p_{1k} + \sum_{k=1}^{Q_2} (Q_1+Q_2-k) \, p_{2k}.$$

It is readily seen that $k_1(x) - k_0(x)$, $x \in I$, depends only on $e(x)$ and is given by (c.f. section 5)

$$(6.6) \qquad k_1(x)-k_0(x) = \begin{cases} L(r_1,r_2+k) & \text{if } e(x) = (r_1,r_2+k), \\ L(r_1+k,r_2) & \text{if } e(x) = (r_1+k,r_2), \end{cases}$$

where

$$(6.7) \qquad L(r_1,r_2+k) = K\delta(Q_1,Q_2-k) + c_{11} \sum_{i=r_1+1}^{r_1+Q_1} \mathcal{E}\{(\underline{\tau}_{1i}-T)\,\iota(\underline{\tau}_{1i}-T)\} +$$

$$+ \sum_{i=r_1+1}^{r_1+Q_1} \mathcal{E}\{(c_{12}+c_{13}(T-\underline{\tau}_{1i}))\;\iota(T-\underline{\tau}_{1i})\} +$$

$$+ c_{21} \sum_{i=r_2+k+1}^{r_2+Q_2} \mathcal{E}\{(\underline{\tau}_{2i}-T)\;\iota(\underline{\tau}_{2i}-T)\} +$$

$$+ \sum_{i=r_2+k+1}^{r_2+Q_2} \mathcal{E}\{(c_{22}+c_{23}(T-\underline{\tau}_{2i}))\;\iota(T-\underline{\tau}_{2i})\}$$

$$\text{for } k = 1, \ldots, Q_2.$$

The formula for $L(r_1+k,r_2)$, $k = 1, \ldots, Q_1$, is obtained from (6.7) by interchanging the indices 1 and 2 in the right-hand member of (6.7).

Using (3.7), (3.8), (3.9), (3.15), (3.18), (3.19) and (3.20), we have

$$\mathcal{E}\{(\underline{\tau}_{jm}-T)\;\iota(\underline{\tau}_{jm}-T)\} = \sum_{h=0}^{m} \frac{(m-h)}{\lambda p_j}\, p(h;\lambda T p_j) =$$

$$= \frac{m}{\lambda p_j} - T + \frac{1}{\lambda p_j} \sum_{h=m+1}^{\infty} P(h;\lambda T p_j), \quad m \geq 1;\ j = 1,\ 2,$$

and

$$\mathcal{E}\{(c_{j2}+c_{j3}(T-\underline{\tau}_{jm}))\;\iota(T-\underline{\tau}_{jm})\} = c_{j2}P(m;\lambda T p_j) +$$

$$+ \frac{c_{j3}}{\lambda p_j} \sum_{h=m+1}^{\infty} P(h;\lambda T p_j), \quad m \geq 1;\ j = 1,\ 2.$$

It is readily seen that the assumptions 3, 4 and 6 are satisfied. From (6.1), (6.2), (6.5), (6.6) and theorem 2.1 it follows now that the long-run average expected cost per unit of time for the (r_1,Q_1,r_2,Q_2) policy is given by

$$(6.8)\qquad g = \frac{1}{\beta}\{\sum_{k=1}^{Q_1} L(r_1+k,r_2)p_{1k} + \sum_{k=1}^{Q_2} L(r_1,r_2+k)p_{2k}\}.$$

Observe that the right-hand member of (6.8) reduces to the right-hand member of (5.11), when we take $p_1 = 1$, $c_{11} = c_1$, $c_{12} = c_2$, $c_{13} = c_3$, $c_{21} = c_{22} = c_{23} = 0$ and $\delta(k_1,k_2) \equiv 1$.

Consider now the symmetric case

$$r_j = r,\ Q_j = Q,\ p_j = \frac{1}{2},\ c_{j1} = c_1,\ c_{j2} = c_2,\ c_{j3} = c_3 \qquad \text{for } j = 1, 2,$$

and suppose

$$\delta(k_1,k_2) \equiv 1.$$

Let

$$p_k = p_{1k}+p_{2k}, \qquad k = 1, \ldots, Q,$$

then

$$(6.9)\qquad p_k = \binom{2Q-k-1}{Q-1}\left(\frac{1}{2}\right)^{2Q-k-1}, \qquad k = 1, \ldots, Q.$$

Let N be a fixed nonnegative integer, and let

$$u_r = \binom{2N-r}{N} 2^{-2N+r}, \qquad r = 0, 1, \ldots, N.$$

The following identities are well known[13, p. 31]

$$(6.10)\qquad \sum_{r=0}^{N} u_r = 1,\ \sigma \overset{\text{def}}{=} \sum_{r=0}^{N} ru_r = (N+1)\binom{2N+1}{N}\ 2^{-2N} - 1,$$

and

$$(6.11)\qquad \sum_{r=1}^{N} \frac{1}{2}\, r(r-1)u_r = N-2\sigma.$$

We note that the probabilities u_r appear in Banach's matchbox problem [4].

Using the identities (6.10) and (6.11), we obtain after some straightforward calculations

$$(6.12)\qquad \sum_{k=1}^{Q} kp_k = 2Q\binom{2Q}{Q}\ 2^{-2Q}$$

and

$$(6.13)\qquad \sum_{k=1}^{Q} k(k+1)p_k = 2Q.$$

By (6.5) we have

$$(6.14)\qquad \beta = \frac{1}{\lambda} \sum_{k=1}^{Q} (2Q-k)p_k = \frac{2Q}{\lambda}\left(1-\binom{2Q}{Q}\ 2^{-2Q}\right).$$

Using (6.12), (6.13) and (6.14) we obtain after some straightforward calculation that the formula (6.8) can be simplified to

$$g = \frac{\lambda}{2Q}\left[1-\binom{2Q}{Q}2^{-2Q}\right]^{-1}\Big[K+\frac{2c_1Q}{\lambda}\left(r-\frac{\lambda}{2}T+\frac{Q+1}{2}\right) +$$

$$+\frac{2(c_1+c_3)}{\lambda}\sum_{i=1}^{Q}\ \sum_{j=r+i+1}^{\infty} P(j;\tfrac{\lambda}{2}T) + c_2 \sum_{i=1}^{Q} P(r+i;\tfrac{\lambda}{2}T) +$$

$$+ c_1\left\{\frac{Q(Q-1)}{\lambda} + \left(\frac{2r}{\lambda} - T\right)\left(Q-2Q\binom{2Q}{Q}2^{-2Q}\right)\right\} +$$

$$+\frac{2(c_1+c_3)}{\lambda}\sum_{k=1}^{Q} p_k \sum_{i=r+k+1}^{r+Q}\ \sum_{j=i+1}^{\infty} P(j;\tfrac{\lambda}{2}T) +$$

$$+ c_2 \sum_{k=1}^{Q} p_k \sum_{i=r+k+1}^{r+Q} P(i;\tfrac{\lambda}{2}T)\Big].$$

When we take $r = 0$ and $T = 0$, this formula reduces to the known formula [15]

$$g = \{\frac{\lambda K}{2Q} + c_1 Q\} \;/\; \{1-\binom{2Q}{Q}2^{-2Q}\}.$$

Finally, we consider another special case of (6.8). Suppose

$$r_1 = r_2 = T = 0, \quad \text{and } \delta(k_1,k_2) \equiv 1.$$

After some straightforward calculations we find that formula (6.8) simplifies to

$$g = \frac{\lambda K + \sum_{j=1}^{2} \frac{c_{j1}}{2p_j} \{Q_j(Q_j+1) - \sum_{k=1}^{Q_j} k(k+1)p_{jk}\}}{\sum_{j=1}^{2} \sum_{k=1}^{Q_j} (Q_1+Q_2-k)p_{jk}} .$$

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